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# THE MOTION OF A SCREW DISLOCATION IN A WEDGE-SHAPED REGION†

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The problem of antiplane deformation for a wedge-shaped region containing a uniformly moving screw dislocation is considered. A general solution of the problem is obtained using Laplace and Kontorovich–Lebedev integral transformations. It is shown during the solution that the method is suitable for a wedge apex angle greater than $\pi$. The limiting case, when the wedge-shaped region is a half-plane, is also considered. For this case the solution can be simplified considerably. © 1999 Elsevier Science Ltd. All rights reserved.

Problems of the unsteady motion of an edge dislocation in a half-plane [1], on the interaction of a moving screw dislocation and a cylindrical inclusion [2], and the problem of the motion of a screw dislocation in a strip [3] have been considered previously.

## 1. FORMULATION OF THE PROBLEM

Consider a wedge-shaped region $-\varphi < \theta < \varphi$ ($\theta$ is the polar angle) which contains a screw dislocation moving uniformly along a ray $\theta = 0$ with velocity $v$. At the boundary of the region the strict closure condition

$$\mathbf{u}_0(r, \theta, z, t)|_{\theta=\pm\varphi} = 0, \ t > 0 \tag{1.1}$$

is given, where $\mathbf{u}_0 = (u_{r0}, u_{\theta 0}, u_{z0})$ is the displacement vector. We will seek a solution in the form

$$\mathbf{u}_0 = \mathbf{u}^{dis} + \mathbf{u}$$

where the displacement vector $\mathbf{u}^{dis}$ describes the motion of the screw dislocation in the plane while $\mathbf{u}$ is the effect of the wedge boundary. The equations describing the motion of the screw dislocation have the form [4]

$$u_r^{dis} = 0, \ u_\theta^{dis} = 0$$

$$u_z^{dis}(r,\theta,t) = \begin{cases} \xi(r,\theta,t) - \eta(r,\theta,t), & 0 < \theta < \pi \\ -(\xi(r,\theta,t) - \eta(r,\theta,t)), & -\pi < \theta < 0 \end{cases} \tag{1.2}$$

$$\xi(r,\theta,t) = \frac{b}{2\pi}\operatorname{arctg}\zeta(r,\theta,t), \ \eta(r,\theta,t) = \frac{b}{2}\mathrm{H}(\zeta(r,\theta,t))$$

$$\zeta(r,\theta,t) = \frac{\gamma r \sin\theta}{r\cos\theta - a - vt}, \ \gamma = \sqrt{1 - \frac{v^2}{c^2}}, \ c = \sqrt{\frac{\mu}{\rho}}$$

where $b$ is the value of the Burgers vector, $c$ is the propagation velocity of transverse waves, $a > 0$ is the distance from the origin of coordinates to the vertex of the dislocation at the instant of time $t = 0$, $H(x)$ is the Heaviside function, while the branch arctg is chosen so that $\operatorname{arctg}(z) \in (-\pi/2, \pi/2)$.

By virtue of boundary conditions (1.1) and formulae (1.2), it is sufficient to determine the third component of the displacement vector $\mathbf{u} = (u_r, u_\theta, u_z)$, since the other two are zero. The required function $u_z = w(r, \theta, t)$ must satisfy the following equation in the region considered

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0 \tag{1.3}$$

$$r > 0, \ t > 0, \ -\varphi < \theta < \varphi$$

and the following conditions on the boundary of the region

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$$w(r,\theta,t)|_{\theta=+\varphi}=-w(r,\theta,t)|_{\theta=-\varphi}=-(\xi(r,\varphi,t)-\eta(r,\varphi,t)) \tag{1.4}$$

In addition, the following condition must be satisfied

$$w(r,\theta,t)\to 0 \text{ as } t\to+\infty \tag{1.5}$$

which denotes that the effect of the dislocation on the boundary of the wedge-shaped region decreases with time.

## 2. SOLUTION

We will seek a solution in the form $w = w^a + w^h$, where the functions $w^a$ and $w^h$ satisfy Eq. (1.3) inside the wedge and the following conditions on the boundary

$$w^a(r,\theta,t)|_{\theta=+\varphi}=-w^a(r,\theta,t)|_{\theta=-\varphi}=f(r,t)=-\xi(r,\varphi,t) \tag{2.1}$$

$$w^h(r,\theta,t)|_{\theta=+\varphi}=-w^h(r,\theta,t)|_{\theta=-\varphi}=\eta(r,\varphi,t) \tag{2.2}$$

It can be shown that the function

$$\hat{w}^a_{\lambda p}=K_{i\lambda}(\frac{p}{c}r)\lambda\,\mathrm{sh}(\pi\lambda)\frac{2}{\pi^2}(A(\lambda,p)\frac{\mathrm{sh}(\lambda\theta)}{\mathrm{sh}(\lambda\varphi)}+B(\lambda,p)\frac{\mathrm{ch}(\lambda\theta)}{\mathrm{ch}(\lambda\varphi)})e^{-pt} \tag{2.3}$$

where $A(\lambda,p)$, $B(\lambda,p)$ are certain unknown functions and $p$ is a real and positive parameter, is a particular solution of Eq. (1.3). Integrating Eq. (2.3) with respect to $\lambda$ and $p$ from 0 to $+\infty$, we obtain a solution of Eq. (1.3), which depends on the derivatives of the functions $A(\lambda,p)$, $B(\lambda,p)$

$$w^a(r,\theta,t)=\int_0^{+\infty}\int_0^{+\infty}\frac{2}{\pi^2}\lambda\,\mathrm{sh}(\pi\lambda)(A(\lambda,p)\frac{\mathrm{sh}(\lambda\theta)}{\mathrm{sh}(\lambda\varphi)}+B(\lambda,p)\frac{\mathrm{ch}(\lambda\theta)}{\mathrm{ch}(\lambda\varphi)})K_{i\lambda}(\frac{p}{c}r)e^{-pt}d\lambda dp \tag{2.4}$$

To determine the functions $A(\lambda,p)$ and $B(\lambda,p)$ we use boundary conditions (2.1)

$$w^a(r,\theta,t)\big|_{\theta=\pm\varphi}=\int_0^{+\infty}\int_0^{+\infty}\frac{2}{\pi^2}\lambda\,\mathrm{sh}(\pi\lambda)(\pm A(\lambda,p)+B(\lambda,p))K_{i\lambda}(\frac{p}{c}r)e^{-pt}d\lambda dp=\pm f(r,t) \tag{2.5}$$

(either the upper or lower signs are taken simultaneously).

Applying an inverse Laplace transformation and a direct Kontorovich–Lebedev transformation to the function $f(r,t)$, we obtain

$$f(r,t)=\int_0^{+\infty}\int_0^{+\infty}\frac{2}{\pi^2}\lambda\,\mathrm{sh}(\pi\lambda)\hat{f}(\lambda,p)K_{i\lambda}(\frac{p}{c}r)e^{-pt}d\lambda dp \tag{2.6}$$

where

$$\hat{f}(\lambda,p)=\int_0^{+\infty}g(r,p)K_{i\lambda}(\frac{p}{c}r)\frac{dr}{r},\quad g(r,p)=\frac{1}{2\pi}\int_{d-i\infty}^{d+i\infty}f(r,t)e^{pt}dt \tag{2.7}$$

Equating the integrands in (2.5) and (2.6), we obtain the unknown functions

$$A(\lambda,p)=\hat{f}(\lambda,p),\quad B(\lambda,p)=0$$

We will now analyse the solution obtained. Applying formula 5.8 (17) of [5] we obtain

$$g(r,p)=\frac{b}{2\pi p}\exp(-\frac{a}{v}p)\exp(\frac{r\cos\varphi}{v}p)\exp(\frac{\gamma r\sin\varphi}{v}p) \tag{2.8}$$

The last factor on the right-hand side increases without limit when $0<\varphi<\pi/2$ as $r\to+\infty$. The Kontorovich–Lebedev transformation cannot be used in this case, and hence we will confine ourselves to considering a wedge with apex angle $\pi/2<\varphi<\pi$. It can be shown that the function $g(r,p)$ satisfies the sufficient conditions for the Kontorovich–Lebedev transformation to be applicable [6]

$$|g(r,p)|r^{-1}\ln\frac{1}{r}\in L_1(0,\frac{1}{2}),\quad |g(r,p)|r^{-1/2}\in L_1(\frac{1}{2},+\infty) \tag{2.9}$$

Note that when $\pi/2 < \varphi < \pi$, by virtue of boundary conditions (2.2), the function $w^h \equiv 0$, and consequently $w = w^p$. We rewrite (2.8) in the form

$$g(r,p) = \frac{b}{4\pi pi}\exp(-\frac{a}{v}p)(\exp(-rz) - \exp(-r\bar{z})),\ z = -\frac{p}{v}(\cos\varphi + i\gamma\sin\varphi)$$

To evaluate integral (2.7) we will use the properties of the Laplace transformation

$$\int_0^{+\infty}\exp(-zr)K_{i\lambda}(\frac{p}{c}r)\frac{dr}{r} = \int_z^{+\infty}\int_0^{+\infty}\exp(-sr)K_{i\lambda}(\frac{p}{c}r)drds \tag{2.10}$$

Using the expression for the Laplace transform for the function $K_{i\lambda}(rp/c)$ (formula 4.16 (24) in [5]) and evaluating the outer integral in (2.10), we obtain

$$\hat{f}(\lambda,p) = \frac{b\pi}{4pi}\frac{1}{\lambda \mathrm{sh}(\pi\lambda)}\exp(-\frac{a}{v}p)(\cos(\lambda \mathrm{arcch}\frac{cz}{v}) - \cos(\lambda \mathrm{arcch}\frac{c\bar{z}}{v}))$$

where $\mathrm{Im}(\mathrm{arcch}\omega) = \mathrm{Im}(\ln(\omega^2 - 1) \in (-\pi, \pi)$.

## 3. THE CASE OF A HALF-PLANE

Let us consider a half-plane as the wedge-shaped region. In this case, boundary conditions (2.1) become

$$w(r,\theta,t)|_{\theta=\pi/2} = -w(r,\theta,t)|_{\theta=-\pi/2} = f(r,t) = -\frac{b}{2\pi}\mathrm{arctg}\frac{\gamma r}{-a-vt}$$

Using the formula for the inverse Laplace transformation for the function $f(r, t)$, we obtain

$$g(r,p) = \frac{b}{2\pi p}\exp(-\frac{a}{v}p)\sin(rp\frac{\gamma}{v}) \tag{3.1}$$

Condition (2.9) is not satisfied for the function $g(r,p)$ but the Kontorovich–Lebedev formula still applies

$$\hat{f}(\lambda,p) = \frac{b}{4p}\exp(-\frac{a}{v}p)\frac{\sin(\lambda s)}{\lambda \mathrm{ch}(\pi\lambda/2)};\ s = \mathrm{arcsh}\frac{c\gamma}{v} > 0$$

Applying an inverse Kontorovich–Lebedev transformation to the solution (formula 12.1(2) of [7]) and using a Laplace transformation (formula 4.5(4) of [5]), we obtain

$$w(r,\theta,t) = \frac{b}{2\pi}\mathrm{arctg}\frac{r\gamma\sin\theta}{r\cos\theta + a + vt} \tag{3.2}$$

For the displacement $u_{z0}(x, y, t)$ we obtain, in a Cartesian system of coordinates

$$u_{z0}(x,y,t) = \frac{b}{2\pi}(\mathrm{arctg}\frac{y\gamma}{x+a+vt} + \mathrm{arctg}\frac{y\gamma}{x-a-vt}) \tag{3.3}$$

The method of reflections gives a similar result. It can be proved that solution (3.3) holds both when $y > 0$ and when $v < 0$.

Thus, when the initial configuration is identical with an elastic half-plane, the stress at the boundary is

$$\sigma_{xz}|_{x=0} = -\frac{\mu b}{2\pi}\frac{2y\gamma}{(y\gamma)^2 + (a+vt)^2}$$

where $\mu$ is the shear modulus. When $v > 0$ the absolute value of $\sigma_{xz}$ falls off as $O(t^{-2})$ when $t \to +\infty$. Hence, if we know the stresses at an arbitrary point on the boundary at different instants of time, we can determine both the direction of a uniformly moving screw dislocation and its position with respect to the point of observation.

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